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An Algebraic Approach to Derivatives

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AN ALGEBRAIC APPROACH TO DERIVATIVES

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AN ALGEBRAIC APPROACH TO DERIVATIVES

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Dr. Reginald Traylor's help was invaluable, especially his introduction to the teachings of R. L. Moore, a distinguished professor of mathematics who taught for many years at the University of Texas at Austin. The hours Dr. Traylor gave in explaining, demonstrating, investigating, calculating, prodding, researching, and encouraging are the force behind the completion of the work. Dr. Ben Fitzpatrick's presentation of the proof of the Pythagorean Theorem and his subsequent contributions in finding the original proof of this argument added color and intrigue to the pursuit of the paper, Dr. Haoxuan Zhou's translation of the original Chinese proof of the Pythagorean Theorem helped to solve the mystery in the writing, and Dr. Cheryl Anderson's clarification of computer problems and technical support in the presentation of this material at various mathematics meetings lent credibility and professionalism to the thesis. Finally, the IWC Education Department's willingness to share its presentation equipment made the delivery of the information colorful and concise for the audiences to whom it was presented.

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CHAPTER I

INTRODUCTION

The purpose of this thesis is to relate a unique method of dealing with the concept of the derivative of a function. The traditional approach to teaching calculus introduces the idea of the limit of a function early in the course as evidenced in the *Essential Elements* of the State Board of Education of Texas which places the limit concept as the second essential element of calculus in Texas high schools (Texas Education Agency (TEA) *Essential Elements* 1991). The "concepts and skills associated with the derivative" (TEA *Essential Elements* 1991) follow immediately after the introduction of the limit of a function and appear to be dependent upon the understanding of the limit. Apparently, finding tangent and normal lines, deriving formulas for the derivative, solving maximum and minimum problems, and finding velocity and acceleration values must be preceded by the development of the concept of a limit (TEA *Essential Elements* 1991)(see Appendix A).

On the college level, changes in calculus curriculum have been tossed around for the past forty years since the Committee on the Undergraduate Program in Mathematics (CUPM) issued a call for revamping in 1953 (Steen 1989). Both Thomas W. Tucker (1982) in his *Priming the Calculus Pump: Innovation and Resources* and Ronald G. Douglas (1986) in *Toward a Lean and Lively Calculus* point out that change is necessary and give details as to the progress toward that change in response to the CUPM call. Tucker (1982) states that "if mathematicians really are tired of high failure rates and low retention of what is taught, then it is time to try something

new." Douglas (1986) admits, "dissatisfaction with teaching calculus and with the results of such teaching has grown to the point that there was unanimity at this conference [the Tulane University Conference/Workshop to Develop Alternative Curriculum and Teaching Methods For Calculus at the College Level, January 2-6, 1986] that something had to be done." The proposal coming from the Tulane Conference suggested the use of graphing calculators and computers and offered a course syllabus which begins with the slope of the tangent line but follows the slope concept with the concept of limit (Douglas 1986). Keeping a fairly traditional approach to the concept of limit, Douglas (1986) concedes, "We recommend a precise english definition of '$\lim f(x) = L$' rather than the mathematically professional $\varepsilon - \delta$ definition." This moves the Tulane Conference recommendations a little further from the usual treatment of the limit while maintaining the conceptual depth of the traditional approach. Indeed, Douglas (1986) cites, "The Conference agreed that the syllabus should be *leaner*, contain *fewer topics*, but that it should have more *conceptual depth*, numerically and geometrically [italics in original quote]." Mathematicians have responded to this call for reform by "getting grants, setting up laboratories, developing software, assigning student projects, writing their own textbooks, and rethinking from top to bottom what should go into a calculus course (Tucker 1982)." However, this author believes there is more that can be done not only to make calculus more precise for the mathematics major but also to make it available and relevant to other fields and to the general public.

Therefore, in order to make the approach to the derivative more concise and accessible, the method of addressing the derivative used in this paper circumvents the notion of a limit and uses only algebra as the vehicle

to mastering the concept of derivatives. This method would enable the student who has no higher mathematics background than high school algebra to understand and apply the concept of derivative. Consequently, this particular manuscript takes on the style of writing appropriate for a textbook addressing:

(1) a business mathematics course,

(2) a mathematics course for the liberal arts major, or

(3) a high school course in analysis.

Part or all of this paper will be submitted to prospective publishers as an example of the possibilities this unique method of dealing with the concept of derivatives provides for producing textbooks.

The presentation of this algebraic approach to derivatives depends heavily upon the definitions for a simple graph, slope of a simple graph, and tangency used by R. L. Moore when he taught at the University of Texas at Austin during the years from 1920 to 1969 (Traylor 1993). These definitions appear in Chapter 2 (Moore 1972)(See Appendix B). Traditional definitions of function (James and James 1992), slope of a line (Lial and Hornsby 1992), The Power Rule for Derivatives (Stewart 1991), and The Binomial Theorem (Streeter, Hutchison and Hoelzle 1991) also appear in Chapter 2. This algebraic/set theoretic approach avoids the notion of a limit; however, the concept of limits could be introduced at the end of a textbook centered around the algebraic approach to the derivative. The benefit of an algebraic approach to derivatives enables any student who has had high school algebra to understand the essentials of calculus. Although many dismiss calculus as much too difficult for the average person to master, this author believes an algebraic approach can simplify one's entry into higher mathematics and enrich day-to-day living with a

broader knowledge of the mathematics which pervade our technical world today.

Technology is an integral part of not only the academic world but almost every profession. This thesis not only involves mathematical research but also necessitated the mastering of the technology used in publishing and in presenting it at various professional meetings. The Equation Editor™ function available in Microsoft Word™ software for the Macintosh™ computer was an invaluable tool in preparing the lengthy algebraic proofs. The arrangement of text and graphics was also made possible by the page layout and desktop publishing features of Microsoft Word™. When presenting the research at such meetings as the Eisenhower Grant Writing Conference in Austin, Texas in February, 1994; the Mathematical Association of America, Texas Section meeting at Texas A&M University in April, 1994; and the Conference for the Advancement of Mathematics Teaching in Houston, Texas in July, 1994; the presentation software, Power Point™, made the presentation visual, graphic and colorful. These presentations were made in cooperation with the Education Department of Incarnate Word College, that provided the equipment furnished to it by Centers for Excellence in Development in Education (CEDE), which is the San Antonio collaborative of Texas Education Agency (TEA) Centers for Professional Development and Technology Grant. The purpose of these presentations was to publicize the results of an Eisenhower Grant program through Incarnate Word College, to announce findings of mathematical research which has significant instructional applications, and to gain reaction from professionals at the high school and college levels concerning the study. The research for this paper also involved searching several remote libraries. This search was done through

the Texas Educators Network by electronic mail and Internet Resources. Because an interactive computer interface would be a possible addition to the publication of any textbook for this material, the author also attended a Hypercard computer workshop at Region 20 Education Service Center.

CHAPTER II

DEFINITIONS

DEFINITION 2.1. A simple graph is a point set such that no vertical line contains two points of it.

DEFINITION 2.2. C is said to be the slope of the simple graph M at the point A if and only if

(1) C is a number and A belongs to M and for every two vertical lines with A between them there is a point of M distinct from A between them and

(2) if l is a line of slope C containing A and α is an acute angle with vertex at A and some point of l in its interior then there exist two vertical lines h and k with A between them such that every point of M between h and k and distinct from A is in the interior either of α or of the angle vertical to α.

DEFINITION 2.3. The line l is said to be tangent to the point set M at the point A if and only if

(1) A belongs to M and every circle with center at A encloses a point of M distinct from A and

(2) if α is an acute angle with vertex at A and some point of l in its interior then there exists a circle J with center at A such that every point of M in the interior of J distinct from A is in the interior of α or of the angle vertical to α.

DEFINITION 2.4. The Power Rule for Derivatives. If $y = f(x) = x^n$ where n is a real number, then $y' = f'(x) = nx^{n-1}$. (The apostrophe used here is just one of several ways to denote the derivative.)

DEFINITION 2.5. A function is a collection of ordered pairs of real numbers, such that no two pairs have the same first element.

DEFINITION 2.6. If points P_1 and P_2 with coordinates (x_1, y_1) and (x_2, y_2) respectively, are any two different points on a line, then the slope of the line (denoted by m) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

DEFINITION 2.7. The Binomial Theorem. For any positive integer n,

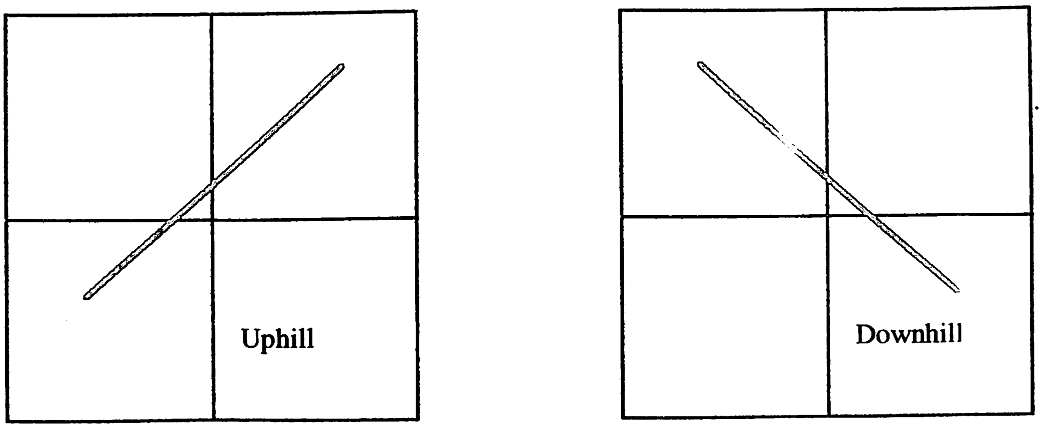
$$(x + y) = x^n + \binom{n}{n-1} x^{n-1} y + \binom{n}{n-2} x^{n-2} y^2 + \binom{n}{n-3} x^{n-3} y^3$$

$$+ \ldots + \binom{n}{n-r} x^{n-r} y^r + \ldots + \binom{n}{1} x y^{n-1} + y^n$$

CHAPTER III

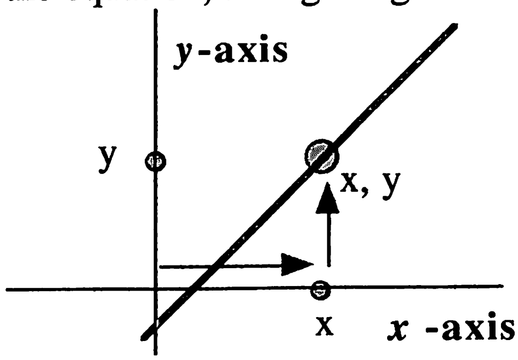
THE IMMUTABILITY OF THE SLOPE OF A LINE

In order to build a bridge from algebra to calculus, a student must turn first to the link between algebra and geometry: the Cartesian coordinate system. This system takes the algebra which was first worked with letters and numbers and gives a "picture" of the equation that has been manipulated. Beginning with the equation of a line, one is able to recognize that some lines travel "uphill" and some travel "downhill" giving each a certain "personality" on the grid system of the Cartesian coordinates.



Although the responsibility for the invention of this system for picturing an equation is uncertain, credit is traditionally divided between René Descartes and Pierre Fermat with Descartes receiving the honor of having his name placed on the system (Streeter, Hutchison, and Hoelzle 1991). In modern usage, the coordinate system places two lines (axes) at right angles to each

other and denotes the horizontal line as the x axis and the vertical line as the y axis. Descartes, however, "used only an x-axis and did not refer to a y-axis," but "for each value of x he computed the corresponding y from the equation, thus getting the coordinates x and y"(Bell 1937).



The algebra student who has been accustomed to dealing with an equation such as, y = 2x - 1, can easily see that substituting different values for x produces unique values for y and results in a pair of "coordinates" which lie on the line pictured as y = 2x - 1. For the value x = 2:

$$y = 2(2) - 1$$
$$y = 4 - 1$$
$$y = 3$$

This generates the pair of coordinates (2,3) on the line y = 2x - 1. For the value x = 7:

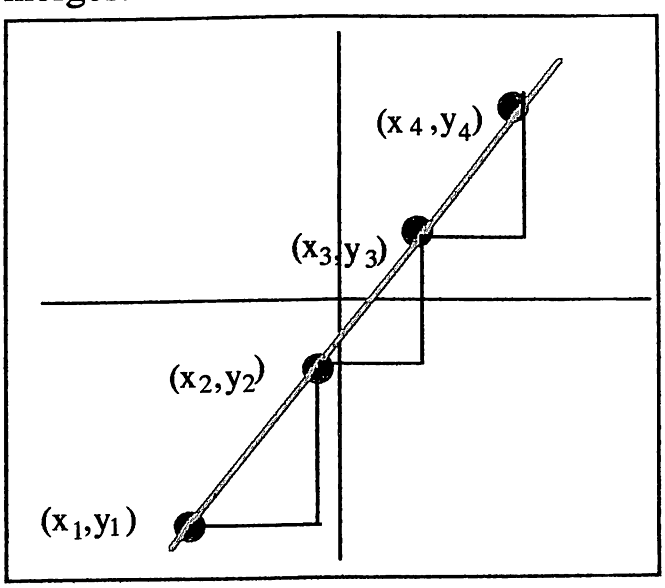
$$y = 2(7) - 1$$
$$y = 14 - 1$$
$$y = 13$$

This produces the coordinates (7,13) on the same line. Using these coordinates, a student is able to measure the distance between any two points on this line. He or she can deal with a function, which is a collection of ordered pairs of real numbers, such that no two pairs have the same first element. And he/she will be able to identify the "personality" or the slope of that line.

An interesting aspect of the slope of a line is that no matter which two points or coordinates of real numbers are chosen on that line, the slope remains the same. A traditional proof of this characteristic of a line involves similar triangles and relies on a fair bit of knowledge of geometry. However, the immutability or unchanging character of the slope of a line can be proved with only the knowledge of algebra. Before leaving an algebra course, most students have become familiar with the general equation for a line which involves coefficients for the *x* and *y* terms plus a constant (TEA *Essential Elements* 1991). This general form for the equation of a line is:

$$\mathbf{ax + by + c = 0}$$

Taking four different values for the coordinates (x,y), the following algebraic proof emerges.



Substituting (x_1, y_1) into the general equation and solving for y_1, results in:

$$ax_1 + by_1 + c = 0$$

$$by_1 + c = -ax_1$$

$$by_1 = -ax_1 - c$$

$$y_1 = \frac{-ax_1 - c}{b}$$

Solving for y using the other values for x and y,

$$y_2 = \frac{-ax_2 - c}{b}$$

$$y_3 = \frac{-ax_3 - c}{b}$$

$$y_4 = \frac{-ax_4 - c}{b}$$

Remembering the definition of the slope of a line:

> *If points P_1 and P_2 with coordinates (x_1, y_1) and (x_2, y_2) respectively, are any two different points on a line, then the slope of the line (denoted by m) is*
>
> $$m = \frac{y_2 - y_1}{x_2 - x_1}$$

and using the values for y calculated above, one may subtract one value of y from the other:

$$y_2 - y_1 = \frac{-ax_2 - c}{b} - \frac{-ax_1 - c}{b}$$

Finding a common denominator of b and combining the numerator, the equation results in:

$$y_2 - y_1 = \frac{-ax_2 - c + ax_1 + c}{b}$$

Combining like terms gives:

$$y_2 - y_1 = \frac{-ax_2 + ax_1}{b}$$

Factoring out a $-a$ leaves:

$$y_2 - y_1 = \frac{-a}{b}(x_2 - x_1)$$

Dividing through by (x_2-x_1) produces:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-a}{b}$$

This equation brings us back to the definition of slope and tells us that the value for the slope is $\frac{-a}{b}$. Using the next values for x and y, the results are as follows:

$$y_3 - y_2 = \frac{-ax_3 - c + ax_2 + c}{b}$$

$$y_3 - y_2 = \frac{-ax_3 - c + ax_2 + c}{b}$$

$$y_3 - y_2 = \frac{-ax_3 + ax_2}{b}$$

$$y_3 - y_2 = \frac{-a}{b}(x_3 - x_2)$$

$$\frac{y_3 - y_2}{x_3 - x_2} = \frac{-a}{b}$$

The last two values for x and y, produce:

$$y_4 - y_3 = \frac{-ax_4 - c + ax_3 + c}{b}$$

$$y_4 - y_3 = \frac{-ax_4 - c + ax_3 + c}{b}$$

$$y_4 - y_3 = \frac{-ax_4 + ax_3}{b}$$

$$y_4 - y_3 = \frac{-a}{b}(x_4 - x_3)$$

$$\frac{y_4 - y_3}{x_4 - x_3} = \frac{-a}{b}$$

Therefore, using just algebra one can prove that the slope of a line remains the same no matter which two points on the line are chosen.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \boxed{\frac{-a}{b}} \qquad \frac{y_3 - y_2}{x_3 - x_2} = \boxed{\frac{-a}{b}} \qquad \frac{y_4 - y_3}{x_4 - x_3} = \boxed{\frac{-a}{b}}$$

Carrying the problem a bit farther, we are able to show that the ratio of the distance from one point to the other is in the same ratio as the ratio between the sides of the triangles in the original "picture" of the line. Taking the ratios above and cross multiplying, we can see that the x sides of the triangle are proportionate and the y sides are proportionate.

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}$$

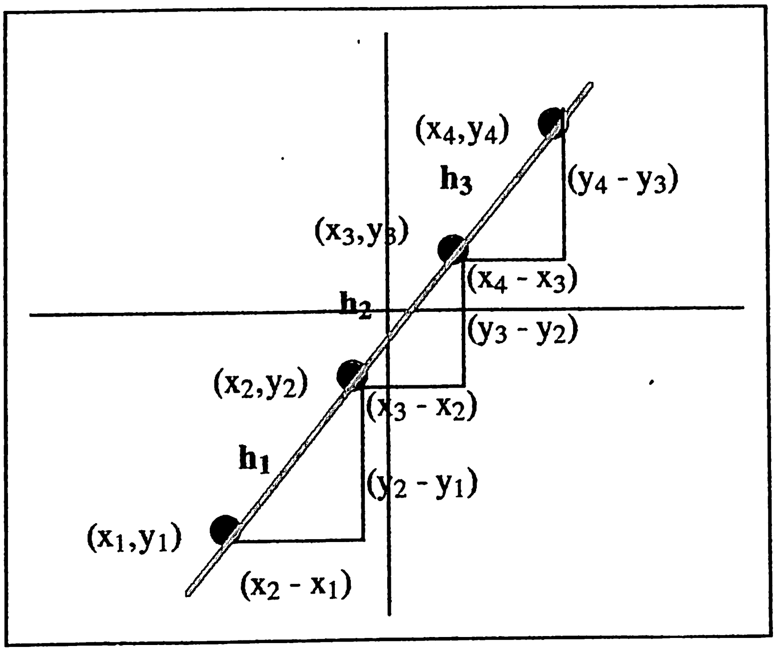
$$(y_3 - y_2)(x_2 - x_1) = (y_2 - y_1)(x_3 - x_2)$$

$$\frac{y_3 - y_2}{y_2 - y_1} = \frac{x_3 - x_2}{x_2 - x_1}$$

$$\frac{y_3 - y_2}{x_3 - x_2} = \frac{y_4 - y_3}{x_4 - x_3}$$

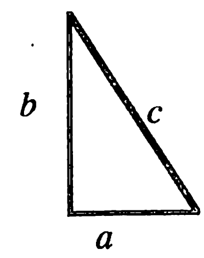
$$(y_4 - y_3)(x_3 - x_2) = (y_3 - y_2)(x_4 - x_3)$$

$$\frac{y_4 - y_3}{y_3 - y_2} = \frac{x_4 - x_3}{x_3 - x_2}$$

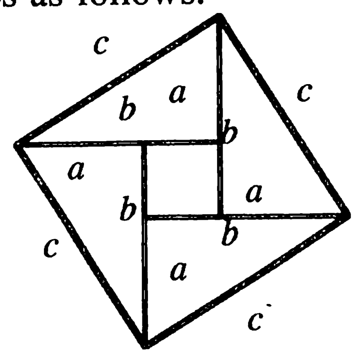


The Pythagorean Theorem would be useful at this point, because distance between each point is actually the hypotenuse of the right triangles formed by the line and the changes in x and the changes in y at each point. Since the *x* and *y* sides to the triangles are proportionate, the Pythagorean Theorem can be used to show that the hypotenuse of each triangle is proportionate to the others. Pythagoras is given credit for the proof showing $c^2 = a^2 + b^2$ with *a* being the shortest leg of a right triangle, *b* being the longer leg, and *c* being the hypotenuse. Frank J. Swetz and T. I. Kao (1977) in their writing, *Was Pythagoras Chinese? An Examination of the Right Triangle Theory in Ancient China* provide a simple proof of Pythagoras' theorem taken from an early Chinese text which dates back to 1100 B.C., 500 years before Pythagoras.

Begin with the right triangle.

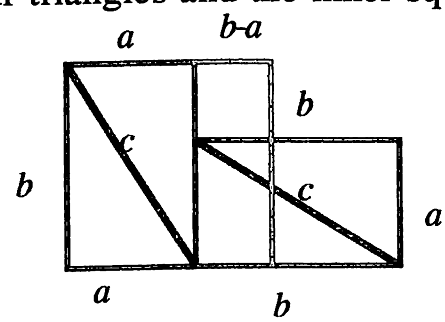


Arrange four such triangles as follows.



The area of the outer square is c^2. Since the long side of the triangle is b, each side of the inner square is b-a.

Now rearrange the four triangles and the inner square as



Now the figure makes up two squares: one with the sides of the length a and the other with sides of length b. The sum of the two squares in this figure is $a^2 + b^2$. Since the area of this figure is the same as the area of the preceding figure, then $c^2 = a^2 + b^2$.

It is quite easy to test this proof by merely taking two 3 x 5 index cards, cutting them diagonally into four right triangles, and arranging and rearranging them as in the above proof. Knowing that $c^2 = a^2 + b^2$, one

may use the following algebraic approach to proving that all the sides of the triangles formed by the line in our picture and the changes of x and y are, in fact, in the same proportions.

Using the Pythagorean Theorem, one can see that the values for h_1 and h_3 are:

$$h_1 = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}\ .$$

$$h_3 = \sqrt{(y_4 - y_3)^2 + (x_4 - x_3)^2}$$

Dividing one hypotenuse by the other results in:

$$\frac{h_1}{h_3} = \frac{\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}}{\sqrt{(y_4 - y_3)^2 + (x_4 - x_3)^2}}$$

Then the question becomes:

$$\frac{h_1}{h_3} = \frac{\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}}{\sqrt{(y_4 - y_3)^2 + (x_4 - x_3)^2}} = \frac{x_2 - x_1}{x_4 - x_3}?$$

Squaring both sides of the equation:

$$\frac{h_1^2}{h_3^2} = \frac{(y_2 - y_1)^2 + (x_2 - x_1)^2}{(y_4 - y_3)^2 + (x_4 - x_3)^2} = \frac{(x_2 - x_1)^2}{(x_4 - x_3)^2}?$$

Cross multiplying gives:

$$(x_4 - x_3)^2\left[(y_2 - y_1)^2 + (x_2 - x_1)^2\right] = (x_2 - x_1)^2\left[(y_4 - y_3)^2 + (x_4 - x_3)^2\right]$$

Distributing produces:

$$\left[(x_4 - x_3)^2(y_2 - y_1)^2\right] + \left[(x_4 - x_3)^2(x_2 - x_1)^2\right] = \left[(x_2 - x_1)^2(y_4 - y_3)^2 + (x_2 - x_1)^2(x_4 - x_3)^2\right]$$

Noticing that $(x_2 - x_1)^2(x_4 - x_3)^2$ appears on both sides of the equation, it can be subtracted from each side leaving:

$$\left[(x_4 - x_3)^2(y_2 - y_1)^2\right] = \left[(x_2 - x_1)^2(y_4 - y_3)^2\right]$$

Dividing through first by $(y_4 - y_3)^2$ then by $(x_4 - x_3)^2$ results in:

$$\frac{(y_2 - y_1)^2}{(y_4 - y_3)^2} = \frac{(x_2 - x_1)^2}{(x_4 - x_3)^2}$$

Taking the square root of both sides answers our question positively.

$$\frac{h_1}{h_3} = \frac{(y_2 - y_1)}{(y_4 - y_3)} = \frac{(x_2 - x_1)}{(x_4 - x_3)}$$

Therefore, we can say that the hypotenuse of each right triangle is in the same proportion to each other as the sides are to one another. This further establishes the fact that the slope of a line is immutablé no matter which two points on that line are used to determine its slope and introduces the concept of similar triangles using only algebra and the Pythagorean Theorem.

CHAPTER IV

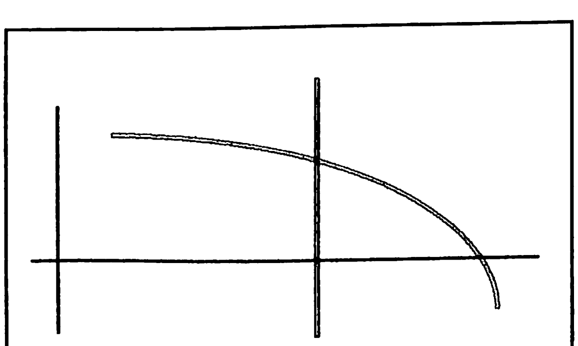
THE SLOPE OF A SIMPLE GRAPH

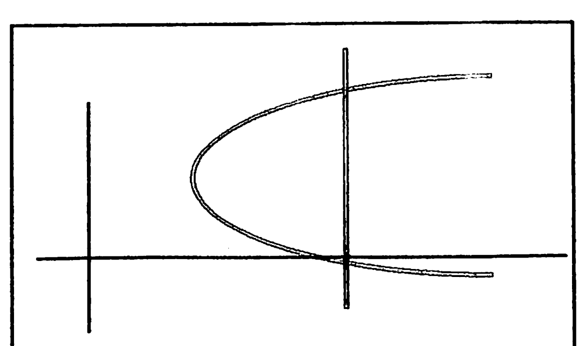
After becoming comfortable with the slope of a line, the natural progression is to move on to the slope of more complicated functions. However, rather than making this jump too complicated, the use of R. L. Moore's definition of a simple graph will ease the transition.

Moore taught at the University of Texas at Austin during the years from 1920 until 1969. His unique method of instruction has become well-known in mathematical circles and has produced numerous outstanding mathematicians (Traylor 1993). Moore's (1972) definition of a simple graph is as follows:

A simple graph is a point set such that no vertical line contains two points of it.

This gives the student a litmus test for deciding if a graph is "simple" by merely imposing a vertical line across it.



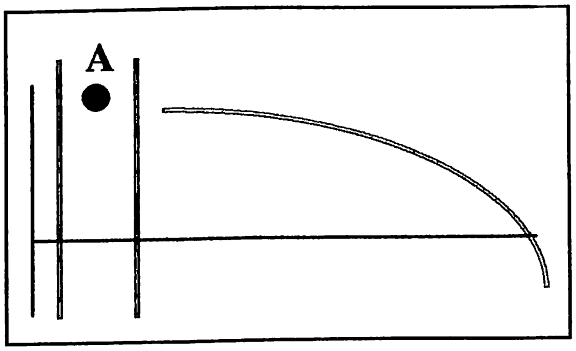


Checking the above graphs with Moore's "test" makes it obvious that the first is a simple graph, but the second fails the test.

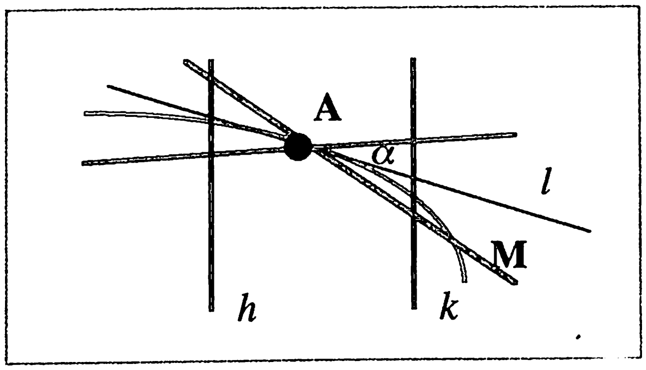
When dealing with the ideas of the slope of a simple graph, once again Moore's (1972) definition is helpful.

> *C is said to be the slope of the simple graph M at the point A if and only if*
>
> *(1) C is a number and A belongs to M and for every two vertical lines with A between them there is a point of M distinct from A between them and*
>
> *(2) if l is a line of slope C containing A and α is an acute angle with vertex at A and some point of l in its interior then there exist two vertical lines h and k with A between them such that every point of M between h and k and distinct from A is in the interior either of α or of the angle vertical to α.*

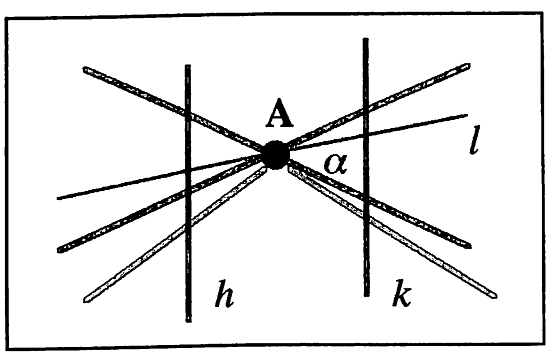
This definition eliminates the instance that A might be an isolated point.



Let A be a point of the simple graph M.



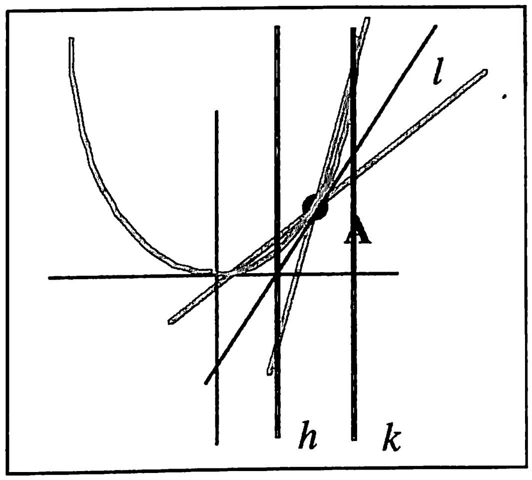
Looking carefully at the above picture of the simple graph M, the line l is the candidate for the slope of the graph. The angle α is any acute angle with vertex at A and some point of l in its interior. The lines h and k with A between are positioned so that every point of M between them and distinct from A is in the interior either of α or the angle vertical to it.



$y = -|x|$

Some graphs do not have a slope line and cannot pass the test for slope of a simple graph. The graph of y = -|x| is clearly a simple graph because no vertical line contains two points of it. However, using the acute angle above, there are no lines h and k, no matter how close they are arranged around A, that can capture this function between the legs of the angle, α, or the angle vertical to it.

Looking at the graph $y = x^2$, it appears to be a simple graph because any vertical line seems to intersect in only one spot. What line would be a candidate as the slope line for this simple graph? Taking l to be the



slope line for the simple graph, $y = x^2$, one must place an acute angle with vertex at some point, A, with l through it and test to see if the graph between two lines h and k remains within that acute angle or the angle vertical to it. It appears as if vertical lines can be drawn to squeeze every point of the graph within those angles and that l indeed could be a slope line for $y = x^2$. If l is such a slope line, then l and $y = x^2$ have one point, namely A, in common and are, therefore, equal at that value for x. Putting the equation of line l in the slope-intercept form of a line, its equation can be written $y = mx + b$ with m being the slope of the line and b being the position of the line's crossing of the y axis. At the point A, $y = mx + b$ is equal to $y = x^2$. Therefore, $x^2 = mx + b$ for the value of x at A. If this equation is solved for the value of m, we will have the slope of the line tangent to x^2.

Solving for m, we have:

$$x^2 = mx + b$$
$$x^2 - mx - b = 0$$

Remembering that the graph and the line have only one point in common, one may assume that the above quadratic has only one root and that it is a perfect square. If this is true, then $x^2 - mx - b = 0$ can be factored into $(x-a)^2$. Using one method of factoring a quadratic that is called "completing the square", we take half the coefficient of x in the equation and subtract it from x making one of the identical factors of a "perfect square" or a quadratic with one root (two identical roots). In the above equation, half of the coefficient of m is $\frac{m}{2}$ which lets us know that the factors of this perfect square are $\left(x - \frac{m}{2}\right)^2$. This means that $a = \frac{m}{2}$, and solving for x indicates that

$$x - \frac{m}{2} = 0$$
$$x = \frac{m}{2}$$
$$2x = m$$
$$m = 2x$$

Therefore, the slope of the simple graph $y = x^2$ is 2x.

Looking at the Power Rule for derivatives: If $y = f(x) = x^n$, where n is a real number, then $f'(x) = nx^{n-1}$, we can see that the slope of the simple graph is actually the derivative of the function. Traditionally, the derivative is taught in most calculus classes by spending several class meetings dealing with the concept of limits and the Binomial Theorem (TEA *Essential Elements* 1991). An algebraic approach, instead of the

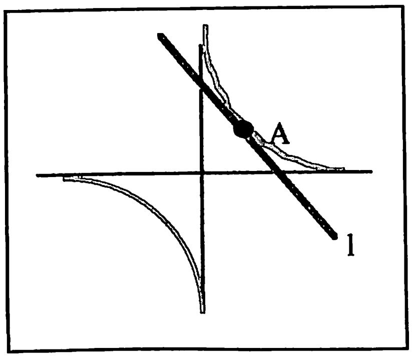
limit method, is relatively painless and can quickly move us into the solution of difficult problems in the real world around us. Taking the derivative of a function can be very useful in for quickly moving us into the solution of difficult problems in the real world around us. Taking the derivative of a function can be applied to maximum or minimum problems, in defining the marginal cost, revenue and profit of a venture, or in any problem in which finding the instantaneous rate of change is necessary.

Although these immediately evident uses of the derivative fall into the realm of the business major, uses for the derivative exist in the liberal arts world also. The point on the graph of a function where the slope of the tangent to that function is zero (horizontal slope) is the maximum or minimum point of the function. An art student wishing to maximize the utility of supplies may set the derivative of the equation of the function describing the variables of his or her project equal to zero and solve for the greatest yield from the supplies available. Once a student sees the usefulness of the derivative, he or she will be able to expand its use in his or her own field.

CHAPTER V

THE SLOPES OF OTHER SIMPLE GRAPHS

By plotting points on the Cartesian coordinate system or by entering the function into a graphing calculator, we can easily come up with a picture of the function, $y = \frac{1}{x}$.



At point A, the line l is equal to the function:

$$mx + b = \frac{1}{x}$$

Multiplying through by x gives us:

$$1 = mx^2 + bx$$

Subtracting 1 from both sides leaves:

$$0 = mx^2 + bx - 1$$

Dividing through by m produces:

$$0 = x^2 + \frac{b}{m}x - \frac{1}{m}$$

Again, using "completing the square", we know that one of the two identical roots of this quadratic is half of the coefficient of x. But we also know that the constant third term is half the coefficient of x squared. In this case, half of the coefficient of x squared gives $\frac{b^2}{4m^2}$, which must be equal to $-\frac{1}{m}$. The square of any number (positive or negative) is positive. Therefore, m must be negative because a negative m times a negative will be positive. It follows that:

$$\frac{b^2}{4m^2} = -\frac{1}{m}$$
$$b^2 = -\frac{4m^2}{m}$$
$$b^2 = -4m$$
$$b = 2\sqrt{-m}$$

This value for b might lead us to believe that our result is an imaginary number; but, consulting the picture of the function and its slope line, we notice that the downhill personality of this line indicates a negative slope. If m is negative then $\sqrt{-m}$ is positive. Now we can substitute the value for b into our factor for the quadratic and solve for m.

$$0 = (x + \frac{b}{2m})^2 = \left(x + \frac{2\sqrt{-m}}{2m}\right)^2 = \left(x + \frac{\sqrt{-m}}{m}\right)^2$$

Setting our factor equal to zero, we have:

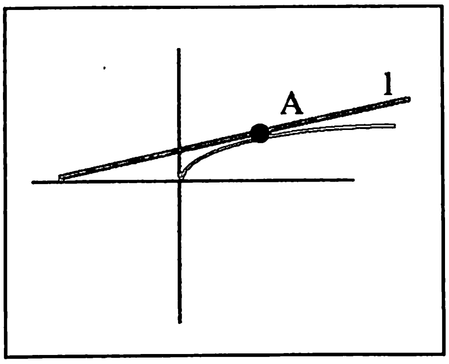
$$x+\frac{\sqrt{-m}}{m}=0$$
$$x=-\frac{\sqrt{-m}}{m}$$
$$x=-\frac{\sqrt{-m}}{\left(\sqrt{-m}\right)\left(\sqrt{-m}\right)}$$
$$x=-\frac{1}{\sqrt{-m}}$$
$$x^2=\frac{1}{-m}$$
$$-mx^2=1$$
$$-m=\frac{1}{x^2}$$
$$m=-\frac{1}{x^2}$$

Remembering The Power Rule for derivatives and re-writing $y=\frac{1}{x}$ as $y=x^{-1}$, we can arrive at the same answer as above.

$$y=x^n$$
$$y'=nx^{n-1}$$
$$y=x^{-1}$$
$$y'=-1x^{-2}$$
$$y'=-\frac{1}{x^2}$$

The derivative of $y=\frac{1}{x}$ is $-\frac{1}{x^2}$, which is the value of the slope of the line tangent to the simple graph of the function.

Graphing the simple graph, $y = \sqrt{x}$, we come up with the following picture of the function:



Setting the equation of the line equal to the equation of the simple graph at the point A, we find that our equation does not look like the ordinary quadratic that we have been working with.

$$mx + b = \sqrt{x}$$

$$mx - \sqrt{x} + b = 0$$

$$x - \frac{\sqrt{x}}{m} + \frac{b}{m} = 0$$

However, since $x = \left(\sqrt{x}\right)^2$, we actually have a first term which is squared just like the normal quadratic. This quadratic also has one root, so we can complete the square and solve for m.

$$\left(\sqrt{x} - \frac{1}{2m}\right)^2$$

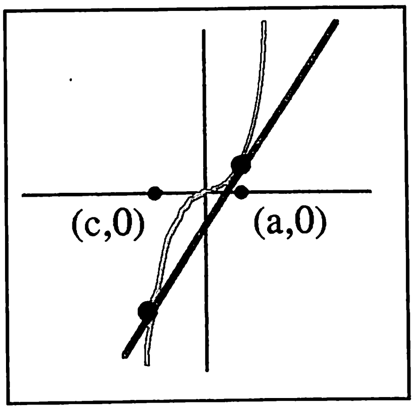
$$\sqrt{x} - \frac{1}{2m} = 0$$

$$\sqrt{x} = \frac{1}{2m}$$

$$m = \frac{1}{2\sqrt{x}}$$

Once again, using The Power Rule for derivatives and remembering that $\sqrt{x} = x^{\frac{1}{2}}$, the derivative of $y = \sqrt{x}$ is $\frac{1}{2\sqrt{x}}$, which is the slope of the line tangent to the simple graph.

Turning to a function involving a higher power, we can investigate $y = x^3$. Looking at the graph of this function, we see:



If we set the function equal to the equation of the line, we have:

$$\begin{aligned}
mx + b &= x^3 \\
-x^3 + mx + b &= 0 \\
x^3 - mx - b &= (x - a)^2 (x - c) \\
x^3 - mx - b &= (x^2 - 2ax + a^2)(x - c) \\
x^3 - mx - b &= x^3 - 2ax^2 - cx^2 + 2acx - a^2 c \\
x^3 - mx - b &= x^3 + (-2a - c)x^2 + (a^2 + 2ac)x - a^2 c
\end{aligned}$$

Since there is no value for x^2 on the left side of the equation, the coefficient of x^2 on the right side must be 0. Therefore, we can say:

$$(-2a - c) = 0$$

$$-2a = c$$

On the left side of the equation, the coefficient of x is $-m$. On the right side of the equation, the coefficient of x is $a^2 + 2ac$. Setting these two values equal to each other, we have

$$-m = a^2 + 2ac$$

If we substitute the above value for c into this, we will get:

$$-m = a^2 + 2a(-2a)$$

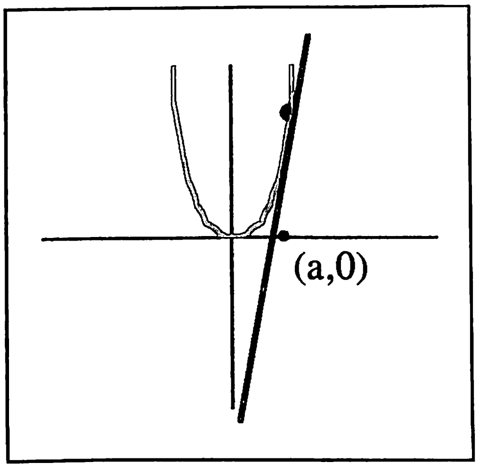
$$-m = a^2 - 4a^2$$

$$-m = -3a^2$$

$$m = 3a^2$$

Once again, using The Power Rule for derivatives, we find that the value of the derivative of the function, $y = x^3$, at a, is $3a^2$. This is the slope of the line tangent to the simple graph at the point a.

If we take $y = x^4$, the graph of the function will look like this:



Setting $y = x^4$ equal to the equation of the line tangent to the function at a, we have:

$$x^4 = mx + b$$

$$x^4 - mx - b = 0$$

$$x^4 - mx - b = (x - a)^4$$

$$x^4 - mx - b = (x - a)^2 (x - a)^2$$

$$x^4 - mx - b = x^4 - 4ax^3 + 6a^2x^2 - 4a^3x + a^4$$

Noting that the coefficient of x on the left side of the equation is $-m$ and the coefficient of x on the right side of the equation is $-4a^3$, we can say:

$$-m = -4a^3$$

$$m = 4a^3$$

Using The Power Rule for derivatives, we can see that

$$y = x^4$$
$$y' = 4x^3$$

Evaluating this derivative at a, it is evident that the slope of the line tangent to the function at a is $4a^3$.

In order to develop a general rule for finding the slope of the line tangent to a simple graph at a specific point, we can use The Binomial Theorem to expand our equation for values of n that are even.

$$x^n = mx + b$$
$$x^n - mx - b = 0$$
$$x^n - mx - b = \left[x + (-a)\right]^n$$

Expanding the right side of this equation using The Binomial Theorem, we will have:

$$x^n + \binom{n}{n-1}x^{n-1}(-a) + \binom{n}{n-2}x^{n-2}(-a)^2 +$$
$$\binom{n}{n-3}x^{n-3}(-a)^3 + \ldots + \binom{n}{n-r}x^{n-r}(-a)^r +$$
$$\ldots + \binom{n}{1}x(-a)^{n-1} + (-a^n)$$

When n and r are non-negative integers, with $r \leq n$, the notation in the elongated parentheses can be re-written:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

The number $n!$ (read "n-factorial") can be defined as follows:

$$n! = n(n-1)(n-2)\ldots(3)(2)(1)$$

A fraction with real, non-negative numbers works out as follows:

$$\left(\frac{6!}{2!}\right)=\left(\frac{6!}{2!(6-2!)}\right)=\left(\frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{(2\cdot1)(4\cdot3\cdot2\cdot1)}\right)=\frac{720}{48}=15$$

From The Binomial Theorem, if we take the x term and set it equal to the x term on the left side of the equation, we will have:

$$-mx=\binom{n}{1}x(-a)^{n-1}$$

In this equation, n is even and n - 1 is odd. Expanding this equation and solving for m, we find:

$$mx=\binom{n}{1}xa^{n-1}$$

$$m=\binom{n}{1}a^{n-1}$$

$$m=na^{n-1}$$

This term of The Binomial Theorem generates the derivative of the function and is equivalent to The Power Rule for derivatives.

This approach can be extended to $f(x)=x^n$ for odd values of n; and, indeed, for all rational functions, $\frac{P(x)}{Q(x)}$, where P and Q are polynomials and Q≠0. With a background of only algebra, a student can grasp the concept of the derivative for any rational function and become familiar with its use in various fields of endeavor. Once the use of the derivative is mastered, a student may then expand his horizons by circling back on the traditional method of dealing with derivatives and investigate the concept of limits. The study of limits will open up the derivatives of trigonometric functions, exponential functions, and logarithmic functions and will become the key to a smooth transition into the world of higher mathematics.

CHAPTER VI
CONCLUSION

This paper introduces a unique approach to the derivative using only algebra and opens the world of calculus to the vast majority of college students and to the public in general. Most, if not all, high school graduates bring with them into the college classroom a knowledge of algebra (TEA *Essential Elements* 1991). Basing the explanation of the derivative on general mathematics knowledge such as the slope of a straight line, the liberal arts major, the business major, or a high school student can master and employ the useful applications of the derivative.

The uses of the derivative are as numerous as there are fields of interests. Once a student understands the concept of the derivative, he or she may then apply this concept to the solution of maximum or minimum problems that present themselves in the fields of study or employment of the individual. Of course, the business student is interested in maximizing profit in business. The derivative of the equation including the variables of his or her venture may be set equal to zero and solved for the maximum profit of the business. The art major may want to cut costs on a project and can, therefore, use the derivative to estimate the maximum number of items cut from a specific amount of matting supply. In the same manner, the fashion design student can estimate yardage of material necessary for a garment just as the agriculture major can predict amounts of fencing needed for storage of a certain number of cattle. However, very few of the

major fields of study require or even suggest the idea of calculus. This could be attributed to the stumbling block of teaching calculus with the use of limits. Few but the mathematics major see beyond this sometimes vague and elusive method of presentation. However, if the student with a high school background in algebra can grasp the concept without the use of limits, the use of the derivative can be opened to other fields and made available to the general public.

With a background of only algebra, a student can grasp the concept of the derivative for any rational function and become familiar with its use in various fields of endeavor. Once a student has the benefit of such an introduction to the idea of the derivative, that student may then expand his or her horizons by circling back on the traditional method of dealing with derivatives and investigate the concept of limits. The study of limits may then open investigation of the derivatives of trigonometric functions, exponential functions, and logarithmic functions and can provide for the student a smooth transition into the world of higher mathematics.

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APPENDIX A

State Board of Education Rules for Curriculum

ESSENTIAL ELEMENTS

After the original free distribution to authorized institutions, additional copies of this publication may be purchased for $24.00 from the Publications Distribution Office, Texas Education Agency. Additional copies are limited to two per request.

Foreword

The *State Board of Education Rules for Curriculum—Essential Elements* presents the essential elements of instruction, the content for the state curriculum which all Texas school districts must teach.

The essential elements have importance because they establish what must be taught in the state curriculum. In addition, they have implications for the statewide assessment program, the content of state-adopted textbooks, and state-level staff development efforts.

In 1981, the 67th Texas Legislature passed House Bill 246 mandating the establishment of essential elements for each elementary subject and secondary course. Today, provisions of this legislation begin with Section 21.101 of the Texas Education Code. The State Board rules for implementing the law are contained in Title 19, Chapter 75 of the Texas Administrative Code.

This publication includes changes in State Board rules through November 1991. Pages are numbered by section. As the Board makes further revisions, the Texas Education Agency will mail replacement pages to school districts, and these pages may be inserted in appropriate sections. Replacement pages will bear the date of Board approval.

In addition to the section page number, each page carries the relevant citation to Title 19, Chapter 75 of the Texas Administrative Code. Below the citation is the Chapter 75 subchapter in which the section appears. Subchapter B comprises the essential elements for prekindergarten-Grade 6; Subchapter C, essential elements for approved courses, Grades 7-8; and Subchapter D, essential elements for approved courses, Grades 9-12.

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(4) Use of inferential statistics to make decisions or to determine validity. The student shall be provided opportunities to:

(A) apply the theory of sampling; and

(B) test hypotheses.

(p) Calculus (1/2-1 unit). Essential elements described in this subsection for calculus shall be superseded by the essential elements described in subsection (cc) of this section effective September 1995. Calculus shall include the following essential elements:

(1) Concepts and skills associated with elementary functions. The student shall be provided opportunities to:

(A) use algebraic, trigonometric, exponential, and logarithmic functions;

(B) investigate properties of functions, including:

(i) sum, product, quotient, composition, and inverse (including domain and range of each);
(ii) absolute value;
(iii) odd and even; and
(iv) zeros of a function.

(C) use fundamental identities including addition formulas for trigonometric functions.

(D) graph functions using:

(i) symmetry;
(ii) asympototes; and
(iii) periodicity and amplitude.

(2) Concepts associated with the limit of a function. The student shall be provided opportunities to:

(A) apply limit theorems and properties;

(B) investigate special limits, including:

(i) the limit defining the Euler number *e*;
(ii) limits involving trigonometric functions; and
(iii) nonexistent limits.

(C) use continuity, including the Intermediate Value Theorem.

(3) Concepts and skills associated with the derivative. The student shall be provided opportunities to:

(A) find tangent and normal lines to a curve;

(B) derive formulas for the derivatives from the limit of a function;

(C) use theorems and properties of the derivative including the relationship between differentiability and continuity:

(D) apply the Mean Value Theorem:

(E) sketch curves using:

(i) techniques for identifying intervals on which the curve is increasing or decreasing;
(ii) relative and absolute maximum and minimum points;
(iii) concavity; and

(iv) points of inflection.

(F) apply concepts and skills, including:

(i) velocity and acceleration;
(ii) related rates;
(iii) maxima and minima; and
(iv) l'Hôpital's rule.

(G) differentiate special functions, such as:

(i) trigonometric functions;
(ii) logarithmic functions; and
(iii) exponential functions.

(4) Concepts and skills associated with the integral and techniques of integration. The student shall be provided opportunities to:

(A) find antiderivatives (indefinite integrals) of functions;

(B) apply antiderivatives to solve problems, including:

(i) distance and velocity from acceleration with initial conditions; and
(ii) differential equations with separable variables including growth and decay.

(C) investigate the definite integral, including:

(i) the concept of the definite integral as an area;
(ii) approximations to the definite integral using rectangles;
(iii) definition of the definite integral as the limit of a sum;
(iv) properties of the definite integral; and
(v) the fundamental theorems of integral calculus.

(D) apply concepts of the definite integral to solve problems involving:

(i) the average value of a function:
(ii) area between curves;
(iii) volume of a solid of revolution (disc, washer, and shell method); and
(iv) length of a curve.

(E) use techniques of integration, including:

(i) basic integration formulas;
(ii) integration by substitution;
(iii) integration by parts; and
(iv) integration by partial fractions.

(F) integrate special functions, including:

(i) trigonometric functions;
(ii) logarithmic functions; and
(iii) exponential functions.

(5) Applications of calculus to special functions.

(q) Number theory (an independent study course—1/2 unit). Number theory shall include the following essential elements:

(A) recognize and use circles and spheres with related parts such as radius, diameter, arc, chord, tangent, secant, and sector;

(B) compute circumferences and areas of circles; and

(C) solve practical problems involving measurements of circles and spheres.

(10) Volume and surface area. The student shall be provided opportunities:

(A) compute the lateral and surface area of common solids;

(B) apply the formulas to practical problems related to areas;

(C) compute the volume of common solids;

(D) apply the formulas to practical problems related to capacity and weight; and

(E) find the surface area or volume of irregularly shaped figures.

(e) **Algebra I (1 unit). Algebra I shall include the following essential elements:**

(1) Comparison of the real number system and its various subsystems in terms of structural characteristics including operations. The student shall be provided opportunities to:

(A) classify real numbers as members of the appropriate subset of real numbers;

(B) identify and use properties of the real numbers;

(C) investigate the density property of real numbers: that between every two real numbers there exists another real number;

(D) investigate the order of operations;

(E) evaluate monomials with integral exponents; and

(F) compare algebraic and geometric definitions of absolute value.

(2) Algebraic representation, solution, and evaluation of problem situations. The student shall be provided opportunities to:

(A) write and evaluate linear expressions from verbal descriptions;

(B) use the properties of equality or models to explain and justify the equation-solving process;

(C) determine the solution to problem situations by writing and solving linear one-variable equations and inequalities;

(D) make a convincing informal argument, orally or in writing, justifying the solution to a problem situation;

(E) solve literal equations for a specified linear variable;

(F) solve systems of equations using linear combinations and substitution as appropriate;

(G) use systems of equations in applications and problem-solving situations; and

(H) solve absolute value equations and inequalities.

(3) Graphing as a tool to interpret linear relations, functions, and inequalities. The student shall be provided opportunities to:

(A) investigate and compare the properties of relations and functions;

(B) describe the domains and ranges of various functions and relations;

(C) identify the relationships among a linear equation, a set of ordered pairs of numbers, and a set of points on a coordinate plane;

(D) explore the concepts of slope and intercept by changing the parameters of a linear equation;

(E) graph a line given characteristics such as two points, one point and slope, table, etc.;

(F) graph a line from its equation in point-slope, general, slope-intercept, or nonstandard forms;

(G) design a statistical experiment to study a problem, recording the results using techniques such as scatter plots, and communicating the outcomes;

(H) write an equation of a line given its graph or description;

(I) use linear equations as models of real-world problem situations;

(J) make predictions from scatter plots that fit linear models;

(K) solve systems of linear equations;

(L) graph linear inequalities with two variables;

(M) graph systems of inequalities; and

(N) explore the relationship between the graph of an absolute value function such as $y = |AX + B| + C$ and the parameters A, B, and C, using computer graphing techniques.

(4) Quadratic equations. The student shall be provided opportunities to:

(A) evaluate quadratic functions for one and for many values of the variable, using a computer or calculator where appropriate;

(B) explore the effects of simple parameter changes on the graphs of quadratic relations, using computer graphing techniques where appropriate;

(C) obtain decimal approximations for the solutions of quadratic equations, using the quadratic formula and a calculator; and

(D) use quadratic equations to make predictions in problem situations.

(5) Polynomials. The student shall be provided opportunities to:

(A) use the definition of polynomial to distinguish between expressions that are polynomials and expressions that are not;

(B) classify polynomials by degree and number of terms;

(C) add, subtract, multiply, and divide polynomials, using concrete models where appropriate;

(D) apply the laws of exponents to include zero and negative integral exponents; and

(E) factor simple polynomials using concrete models where appropriate.

(6) Rational expressions. The student shall be provided opportunities to:

(A) evaluate rational expressions, avoiding division by zero;

(B) apply operations on simple rational expressions (linear or monomial numerators and denominators only);

(C) solve rational equations with linear numerators and denominators;

(D) solve problem situations using ratio and proportion;

(E) use the definition of probability as a ratio of numbers of outcomes to solve problems involving uncertainty;

(F) apply the concept of dimensional analysis (carrying units throughout a computation) in problem situations to determine appropriate units for denominate numbers; and

(G) perform operations on numbers in scientific notation, both mentally and by calculator, and use these numbers in problem situations.

(7) Properties of and operations with square roots. The student shall be provided opportunities to:

(A) use the calculator to approximate numeric radical expressions involving square roots;

(B) simplify algebraic radical expressions involving square roots;

(C) add, subtract, multiply, and divide numeric and algebraic radical expressions involving square roots;

(D) solve simple radical equations involving square roots; and

(E) use the Pythagorean Theorem in problem situations.

(f) **Algebra II (1 unit). Algebra II shall include the following essential elements:**

(1) Development of mathematical structure. The student shall be provided opportunities to:

(A) compare and contrast the real number system and its various subsystems in terms of structural characteristics;

(B) investigate examples and nonexamples of fields using the real number system and its various finite and infinite subsystems; and

(C) develop the complex number system and its operations.

(2) Quadratic functions. The student shall be provided opportunities to:

(A) solve quadratic equations by completing the square;

(B) develop and apply the quadratic formula;

(C) find a quadratic equation given its roots;

(D) explore the effects of simple parameter changes on the graph of a quadratic function, using computer graphing techniques where appropriate;

(E) use characteristics of a quadratic function to sketch the related curve;

(F) determine the equation of quadratic functions from their graphs; and

(G) use quadratic functions as models in real-world problem situations.

(3) Quadratic relations. The student shall be provided opportunities to:

(A) explore the graphs of algebraic representations of conic sections and make generalizations that allow classification of these algebraic representations as circles, ellipses, hyperbolas, or parabolas, using calculators or computers where appropriate;

(B) verify graphs of conic sections using computer graphing techniques where appropriate;

(C) use characteristics of conic sections to sketch the related curves;

(D) determine equations of conic sections from their graphs; and

(E) use quadratic relations as models in real-world problem situations.

(4) Systems of equations. The student shall be provided opportunities to:

(A) use the linear combination (addition-subtraction) method to solve systems of three linear equations in three variables;

(B) use augmented matrices by hand or by computer to solve two- or three-variable linear systems;

(C) apply linear programming techniques to model and solve real-world situations, using the computer or calculator, where appropriate; and

(D) solve quadratic-quadratic and quadratic-linear systems, and confirm the solution by computer graphing techniques.

(5) Numerical methods and higher degree polynomials. The student shall be provided opportunities to:

(A) use successive approximations on the calculator or computer to solve higher degree equations;

(B) apply synthetic substitution to find functional values of higher degree polynomials;

(C) use the Fundamental Theorem of Algebra and the Factor Theorem to factor higher degree polynomials;

(D) graph higher degree polynomial functions using computer graphing techniques;

(E) solve higher degree polynomial equations using computer graphing techniques; and

(F) use an iterative process (algebraic or geometric) to approximate irrational roots of higher degree functions.

(6) Exponential and logarithmic functions. The student shall be provided opportunities to:

(A) investigate the concept of nth root and convert between exponential and radical forms of an expression;

(B) extend the properties of exponents to include rational exponents;

(C) investigate exponential functions and their inverses to develop the definition of logarithm;

(D) explore the graphs of exponential and logarithmic functions using computer graphing techniques;

(E) convert between logarithmic and exponential forms of an equation;

(F) apply properties of logarithms to solve equations; and

(G) apply logarithmic and exponential functions in problem situations using the computer or calculator.

(7) Rational algebraic functions. The student shall be provided opportunities to:

(A) simplify complex fractions;

(B) graph rational algebraic functions (using computer graphing techniques where appropriate) to develop an intuitive understanding of the concept of limit; and

(C) use direct and inverse variation functions as models to make predictions in real-world situations.

(8) Sequences and series. The student shall be provided opportunities to:

(A) investigate patterns in given sequences and use the patterns or recursive or generator formulas to find additional terms;

(B) investigate and graph geometric and arithmetic sequences;

(C) find the nth partial sum of geometric or arithmetic series and find n given the nth term or partial sum;

(D) investigate convergent geometric series;

(E) use sequences and series as models in real-world problem situations;

(F) use the Binomial Theorem to expand powers of binomial expressions; and

(G) solve enumeration problems involving permutations and combinations.

(9) Data handling and analysis. The student shall be provided opportunities to:

(A) recognize the importance of unbiased sampling and valid reasoning in statistical arguments;

(B) select an appropriate sampling method for a given real-world problem situation;

(C) interpret probabilities relative to the normal distribution;

(D) design a simple statistical experiment to test a hypothesis generated by a real-world problem situation and interpret the results; and

(E) use computer simulation methods to represent and solve problem situations involving uncertainty.

(g) Geometry (1 unit). Geometry shall include the following essential elements:

(1) Axiomatic systems. The student shall be provided opportunities to:

(A) distinguish intuitively between the concepts of validity of an argument and truth of a statement;

(B) distinguish between inductive and deductive reasoning;

(C) use conditional statements in logical arguments;

(D) investigate the relationship among a conditional statement and its converse, inverse, and contrapositive;

(E) identify patterns of inference that produce valid conclusions and apply in real-world situations;

(F) apply logical arguments to geometric problem situations; and

APPENDIX B

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Name: JULIA S. Roman

Date: 5-31-94

Telephone: 210-829-3171

APPENDIX B

Ms. for that Film?

"Many years ago, at The University of Pennsylvania, I had a class of three students in a graduate course called Foundations of Mathematics. I stated some axioms and theorems and proposed that they prove the theorems from the axioms. Sometime in December I stated Theorem 15. J. R. Kline presented a proof in the following April. Another member of the class did not want to give up and listen to Kline's proof after he had tried for so long to get a proof of his own and he walked out of the classroom. Towards the

end of the course he said he thought he could prove this theorem if he could only "get off of the boundary" (as he expressed it). More than 12 years later when I was on a visit to a University where he had become a professor, I said "have you got Theorem 15 yet?" and he answered "No."

I am still giving a course bearing the same title but it is based on quite a different set of axioms and a theorem corresponding to the old Theorem 15 now comes much later in the treatment. The number of this course is 688

Axiom 0 states that every region is a point set and Axiom 1 now states that there exists a sequence

satisfying certain conditions numbered 1, 2, 3 and 4

~~$G_1, G_2, G_3, \ldots$ satisfying four conditions.~~

A large body of theorems can be derived from Axioms 0 and I without use of the fourth of these conditions and in 688 I usually do not state the fourth one until some months have gone by. In 1958–59, when the time finally arrived to state it, it appeared that there was one member of the class, a Mr. W., who did not want to ~~know~~ be told what it was. So the others were told but he was not. One day about 2 years later, in my 690 class, he was at the board, explaining something and I started to make a remark. He seized the door knob, flung the door open and rushed out of the room. He thought I

was about to ~~tell him what~~ state Condition 4 condition ~~4 was~~, ~~though I had no such intention~~. I imagine some of you are thinking that it was ridiculous for him to be so ~~anxious~~ determined to remain ignorant concerning this condition. If so I do not agree with you at all. In course of time he thought of quite a different condition such that if it is substituted in place of Condition 4 in the statement of Axiom 1, the resulting axiom (Axiom 1_W) is quite interesting and quite different from Axiom 1. If he had been told at the outset what Condition 4 was he would, I believe, never have thought of Axiom 1_W and never have discovered the interesting things he has proved to be true concerning it.

Often when I have raised some difficult question in one of my graduate classes and days (or weeks or possibly even months) have gone by without its having been settled in that class and finally the day arrives when someone announces that he has a solution and he goes to the board to present it, some of the others members of the class (sometimes most of them) walk out and stay out till he is through. Is it a good thing for a student to walk out under such circumstances? I think that often it is ~~such~~ but sometimes ~~often~~ it is not and in the case of some students it is very hard ~~it~~ for me to decide whether or not to discourage it.

I do not believe I ever said anything to discourage it in the case of Mr. W. I ~~doubt whether~~ don't believe there was a single instance where he walked out when it would have been better for him had he staid in.

~~I was~~ ~~Once~~ One year I had in my 688 class ~~In 1963-64 class in 688~~, a student whom I will call Mr. X and whom I considered to be one of the very best, and probably in the class the very youngest. ~~seemed to be~~ He seemed to be walking out or even completely missing class too often and one day when he was present I made some remarks indicating that this sort of thing could be overdone. After class he came

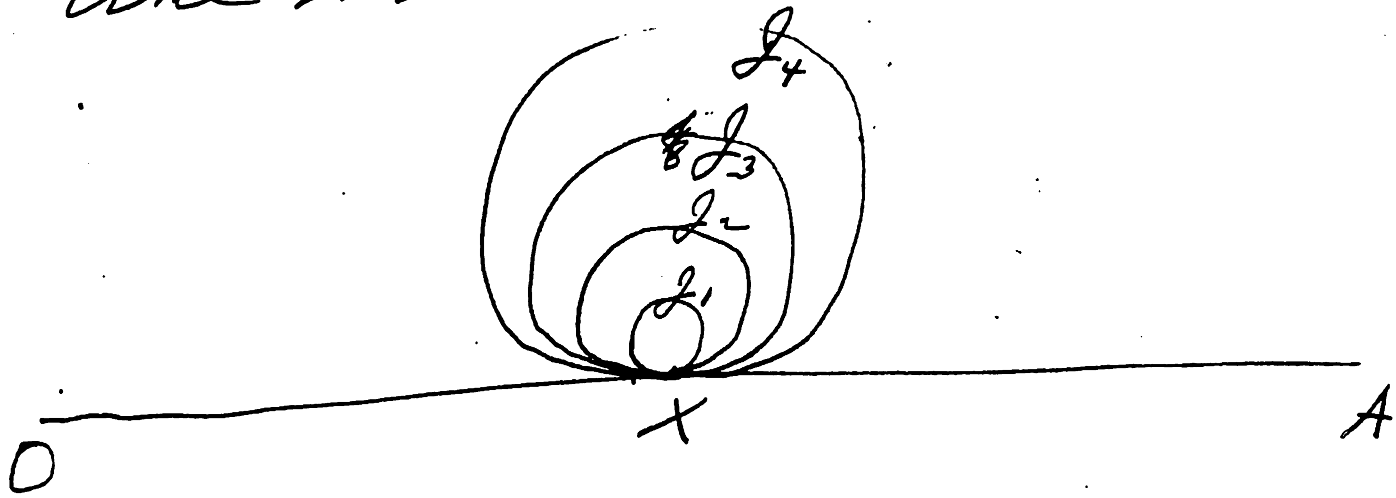
to my office and assured me that whenever he had stayed out when a theorem was being proved he had eventually obtained a proof of his own. After this he proceeded to stay away some more. One day when he was absent a member of the class gave a simple proof of Theorem 70,
theorem to the effect that if A, B and O are three points of a connected point set M and no point separates O either from A or from B in M then no point other than O separates A from B in M. When he was through I said something like this "What would you say if someone would argue this way: Since O is not separated

either from A or from B, in M by any one point therefore if X is a point of $M-(O+A+B)$, $M-X$ contains a connected point set containing O and A and another one containing O and B and, since these two connected point sets have the point O in common, their sum is a connected subset of $M-X$ containing both A and B and therefore A and B are not separated from each other in M by X"? Most of the class ~~agreed~~ seemed to agree that ~~seemed to think~~ that would be a good argument but a few seemed to be in doubt. I asked a ~~one~~ student whom I will call Mr. G. whether at the next meeting of the class attended by Mr. X he would be willing

to give this argument for Theorem 70 and see whether Mr. X would offer any objection to it. He agreed to do so and when the proper time arrived I called on Mr. G to prove Theorem 70. He went to the board and I noticed that Mr. X ~~was~~ did not appear to be listening to him. I said "Are you listening to this argument? Listen to it. Listen carefully. When Mr. G was through I asked X whether G's argument was all right. He said yes, I think so. You don't know that it is? Isn't it true that if every connected subset of M that contains O and A contains X

then X separates O from A in M?
He replied that it was. If you can't go from Austin to San Antonio without going through Buda then doesn't Buda separate Austin from San Antonio? He replied in the affirmative. I said "If you had staid out when Theorem 70 was being proved in class and you had subsequently thought of this argument then would you have felt that here was a case where you had stayed away when a theorem was being proved but you had eventually proved it yourself? He said "Yes." I asked whether there was anyone present who could show Mr. X that if he

had thought so he would have been mistaken. Mr. P went to the board and drew a figure like this:



Here, for each positive integer n, J_n is a circle, of radius n, tangent to the line OA at the point X. If M denotes the line OA plus the sum of all such circles then neither X nor any other point separates O from A in M. But there is no connected subset of $M-X$ containing O and A. I asked Mr. X whether there may not

have been cases where he thought he had proved a theorem that he had not proved. I indicated to him that by staying out a student may miss discussions on occasions when it would have been better for him to have stayed in and taken part in them.

Some years ago I had in 688 a student (Mr. S) of outstanding ability who, I think, seldom, if ever, had any occasion to walk out. Seldom, if ever, did any other student present, in this class, a proof of a theorem that S had not already succeeded in proving. One of the other

members of this class said that
being in a class with S was
like being in a lecture course
(with S as lecturer) and he
did not like lecture courses
so he did not want to take
689 (a sequel to 688) the
next long session because S would
be taking it then. He asked whether
if he and another student should
stay away from the University of
Texas the whole of the next
long session they could count
on my giving 689 again in the
long session immediately following
the next one. I indicated that
I expected to do so and they both
got positions ~~out of Tex~~ ~~in California~~,
stayed away from Texas one session and

returned later according to plan. I think that under the circumstances their decision to stay away was a good one. They were both good students and went on to receive Ph.D. degrees.

I have sometimes said to a student "If you don't want to listen to someone else proving a theorem that you have not yet proved then how about trying to be the first one to prove it and if you fail to be then paying the penalty for not being by staying in and listening to someone else's proof?"

On Tuesday March 10, 1964 I stated Theorem 55 to my class in 688. This theorem is as follows: If H is a countable collection of closed and compact point sets such that no nondegenerate component of an element of H intersects another element of H then the sum of all the elements of H is not a locally compact continuum. This class met on Tuesday, Thursday and Saturday. I received no proof of Theorem 55 on March 12 or March 14 but after class on March 14 Mr. G indicated that he had what he had thought was a counter

I asked whether anyone could prove Theorem 5.5. Mr. G said "I have a counterexample." #. He went to the board and

example but discovered later that it was not one. Without knowing what his example was I suggested that he describe it at the next meeting of the class without disclosing that it is defective. On Tuesday, March 17 ~~he~~ described a certain infinite collection H of mutually exclusive intervals lying on an interval M and ~~so~~ indicated that M is the sum of all the intervals of the collection H, contrary to Theorem 5.5. Two members of this class of 15 had had 624 and were acquainted with an example resembling this one and two others had had work in point set theory elsewhere but I asked each of the remaining eleven except Mr. G whether this ~~seemed to be~~ was a counterexample to Theorem 5.5 and they all replied in the affirmative. (or seemed to be:) I said "Well you now have a choice. Either show there is nothing wrong with this example or prove Theorem 5.5.

On Thursday, March 19, I called on Mr. X and he indicated that he thought he could prove that there is something wrong with this example and he could also prove Theorem 5-5. He went to the board and gave what I think was quite a good argument to show that the sum of the interiors of Mr. S's collection H is not the interior ~~of~~ M so that this is not a counter example to Theorem 5-5. But the problem of proving Theorem 5-5 remained. At the last meeting of the class before the examination I asked whether anyone could prove it. Mr. X raised his hand. Mr. Z went out. The others remained. Mr X went to the board and his argument seemed to be

proceeding nicely till he got to the point where, concerning a certain pair of elements of H he said let T denote an irreducible subcontinuum of M from one of them to the other one. The question was raised whether he was sure that there is such a subcontinuum of M. What if these two elements of H intersect? He appeared uncertain as to how to go on, the bell rang and we adjourned. Mr Z has given me what is supposed to be a written proof of Theorem 5-5 but I have not checked it ~~yet~~.

I am guessing that, at the first meeting of 689 next September, either Mr. X or Mr. Z will prove Theorem 5.5.

Years ago, a course in advanced Calculus when given by me was ~~so~~ usually so different from courses with the same title given by others that I decided to use a different title and a different number. About 1941 the title was changed to Introduction to the Foundation of Analysis and the number was changed to 24 and later to 624.

I do not usually allow in 624 any student who has taken or is taking Advanced Calculus 321 or Differential Equations 322. A student who has had one of these courses is likely

to know in advance the answers to so many of the questions raised in 624 that I would be inclined to call on him in that course either seldom or not at all. One of the questions that I ask, usually near the beginning, in 624 is whether or not there exists, on the X-axis, a closed and bounded point set M such that each point of M is a limit point of M but M contains no interval. On one occasion, sometime after I had raised this question, a student indicated that he could answer it and went to the board. I don't think he had uttered more than 2 or 3 sentences before I became suspicious, stopped him and asked him whether he had read anything on this subject. He replied

that he had not but that he had talked to some one about it. I said "Well, that's enough! You have spoiled this question for this class" and he sat down. It was a long time before I called on him again about anything. In the hall after this class period another student said "He certainly did spoil this question. After he said what he did it was easy to see the answer to the question."

For many years in 624 (and also in my section of calculus) I have defined a simple graph in the plane as a point set such that no vertical line contains two points of it and have pointed out

that (1) no use of the notion function is involved in this definition but (2) the point set M is a simple graph if and only if it is the graph of some function.

The simple graph M is said to be continuous at the point A if and only if A belongs to M and for every two horizontal lines α and β with A between them there exist two vertical lines h and k with A between them such that every point of M between h and k is also between α and β.

C is said to be the slope of the simple graph M at the point A if and only if (1) C is a number and A belongs to M and for every

two vertical lines with A between them there is a point of M distinct from A between them and (2) if l is a line of slope c containing A and α is an acute angle with vertex at A and some point of l in its interior then there exist two vertical lines h and k with A between them such that every point of M between h and k and distinct from A is in the interior either of α or of the the angle vertical to α.

The line l is said to be tangent to the point set M at the point A if and only if (1) A belongs to M and every circle with center at A encloses ~~a~~ a point of M distinct from A and (2) if α

is an acute angle with vertex at A and some point of l in its interior then there exists a circle J with center at A such that every point of M in the interior of J distinct from A is in the interior of α or of the angle vertical to α.

Questions are asked concerning these properties and relationships between them. For example is it true that if a simple graph has a tangent at the point A then it has a slope at that point? If someone gives an example to show that that is not true then ~~what is~~ can be thought of some way to strengthen the hypothesis ~~in order~~ ~~the hypothesis is strengthened by the~~ so ~~that~~ the conclusion would follow ~~supposition that M is continuous at A~~?

Is it true that if the projection of the simple graph M onto the X-axis is closed and M is closed then M is

continuous? If some one gives an example to show that that is not true then the conclusion would follow if the hypothesis were strengthened in what way? An example of what not to do would be to tell the class at the outset to prove that if M is a simple graph whose projection onto the X-axis is closed and bounded then M itself is closed and bounded if and only if it is continuous — to tell them that at the outset and thereby deprive them of the opportunity to both think of it and prove it. As a consequence of this theorem and the theorem that for every closed and bounded pointset M there is a point of M

such that no point of M is higher than it it follows that if M is a simple continuous graph whose projection onto the X-axis is closed there is a point of M with no point of M higher than it.

I have often told a class to prove something that I know is not true for example to prove that if a point set is closed so is its projection onto the X-axis. Isn't this much better than to tell them to prove that the projection onto the X-axis of a closed and bounded point set is closed? Why should any teacher want to follow the latter procedure and therefore deprive a student of the opportunity to discover independantly that one of these propositions is true and the other one is false?

But propositions, true or false, are not the only things to be considered. If it is granted that it is better not to prove or disprove a proposition for a student without at least giving him an opportunity to prove or disprove it for himself then what about concepts and definitions?

Suppose you would like to know whether there is anyone in a certain class who is capable of thinking of a certain concept, thinking of it without the help of any hint whatsoever. Suppose you know of a problem that you believe no one could solve without thinking of and using that concept. Then how about

first carefully avoiding any reference to the concept or anything too closely related to it until you are ready to propose the problem and <u>then</u> proposing it.

This year, as I have done every year for a long time, I raised with my 624 class the question whether, in the plane, there exists a collection Q of closed and bounded point sets such that (1) each element of Q has at least three points, (2) if x and y are elements of Q and x is non degenerate then y is the image of x under some continuous transformation and (3) if x belongs

to Q and y is the image of x under some continuous transformation then y belongs to Q.

Wondering whether some member of the class might think of and introduce the concept connectedness in order to solve this problem, I had purposely refrained from saying anything whatsoever to this class about connectedness at any time before the day in the second semester when I ~~stated raised~~ stated the problem. On April 10, Miss S indicated that if Q is such a collection then no element of Q is the sum of two mutually exclusive straight line intervals — — For suppose an element of Q is

the sum of two such intervals α and β. Let A and B denote two points. There is a continuous transformation throwing α into A and β into B. Therefore, by stipulation (3) concerning Q, $A+B$ belongs to Q. Hence, by (2), $\alpha+\beta$ is the image of $A+B$ under some continuous transformation. But this is impossible ~~impossible not true~~. I suggested that she try to find some other point sets that do not belong to Q. On April 13 she said "If there exists a distance between two subsets of M such that there is no point of M in that distance" then M does not belong to Q. She indicated that ~~she~~ "there is no point of M in that distance" left something to be ~~desired~~ desired but she was trying to get words to describe a point set like — — which she wanted to call separated. I would

much rather have a concept introduced this way by a student naturally in the course of an investigation even if (I am tempted to say *especially* if) it is not perfectly stated at first. I suggested that she continue to think about it. On April 15 she stated a definition of what she called a separated point set which is satisfied if and only if that set is not connected according to Lennes' definition.

Sometimes when a student gives a long argument to prove a theorem and I know of a much shorter one I do not tell him there is a shorter one

fill in

A mathematician from Europe once said to me Oh if you know a shorter ~~method~~ proof you should tell him. If you don't you are not teaching.

I am tempted to paraphrase an often quoted ~~st~~ saying about governments by saying "that student is taught the best who is told the least."